Indispensable Manual Calculation Skills in a CAS Environment

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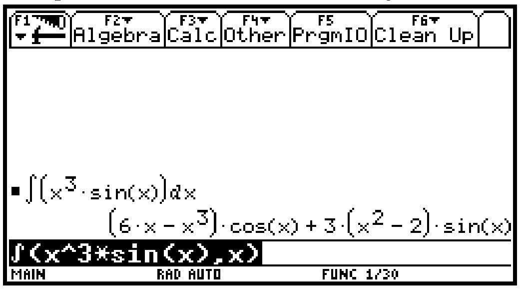
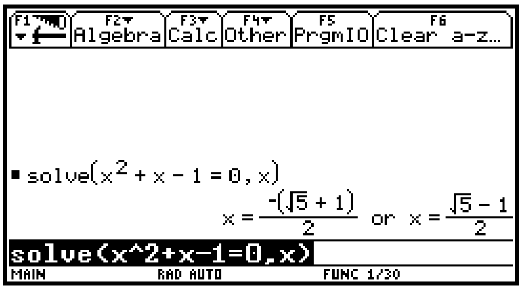
<u>Abstract</u>: Which manual calculation skills are still needed when students use graphic/symbolic calculators or computers with computer algebra systems (CAS)? What should students be able to do manually, i.e. just using paper and pencil? This text is the outcome of a two-day discussion on these questions, held by the four authors. Our answers and proposals are meant to be a challenge, aiming at sparking off a broad discussion about which permanently available manual calculation skills we still need to teach and assess.

Computer algebra systems (CAS)

Computer algebra systems (CAS) are tools which automate the execution of algebraic computations. CAS can simplify expressions, compute symbolic derivatives and integrals, plot graphs, solve equations and systems of equations, manipulate matrices, etc. In short: they automate most of the calculation skills we teach in school mathematics.





The CAS which are widely used in schools are the computer program Derive and the algebraic calculators TI-92 and TI-89. Introductions to using these tools are [Kutzler&Kokol-Voljc 2000] for Derive 5, [Kutzler 1997] for the TI-92 and [Kutzler 1998] for the TI-89.

Soon such tools will be used as a matter of course, such as we use scientific (in some countries also graphic) calculators today. Using a calculator for differentiating $x^3 \sin^2(4x+5)$ will be as common as using it to evaluate $\cos(1.3786)$ or $\sqrt{5.67}$. The above screen images give examples of what CAS can do.

Starting Point: Calculator-Free Exam

We assume an exam comprising two parts. In the first part no modern technical tool is permitted – not even a simple scientific calculator – whereas in the second part all kinds of technology,[1] in particular powerful calculators or computers with CAS may be used. Some countries, such as Austria, are experimenting with two-tier exams. Other countries, such as England, use two-tier exams already. We believe that two-tier exams would be a well-balanced compromise meeting both the desires of technology supporters and the reservations of those who are concerned about

[1] It would be better to use the word "calculator" here. According to *The CASSELL Encyclopaedia* the word "technology" means "the science of the industrial arts; the practical application of science to industry and other fields; the total technical means and skills available to a particular human society; the terminology of an art or science." However, the use of the word "technology" became ingrained in the academic literature about the use of calculators and computers for teaching. Therefore – and because we want to

the use of technology in the classroom. Some fundamental thoughts about two-tier exams are contained in [Kutzler 1999].

We assume a fictitious, written, technology-free exam. We look for questions and classes of questions which we would include in such an exam.

Drawing the border line between questions to be asked in a technology-free exam and questions which would not be asked in such an exam is equivalent to listing the indispensable manual calculation skills. Therefore, the fictitious technology-free exam is a means to an end for us. Our discussion and its results are relevant far beyond the exam situation. They are fundamental for the development of mathematics education in the years to come.

After reconsidering the meaning and importance of calculation skills and restraining their role in teaching and learning, it is crucial to discuss the consequences for mathematics teaching. This will become the topic for our future discussions and work.

Three Pots: -T, ?T, +T

The border line between questions to be asked in a technology-free exam and questions not to be asked in such an exam clearly depends on many parameters – including the type of school. We try to give a universally applicable answer by creating three pots, which we name –T, ?T, and +T.

- The first pot, –T (= no technology), contains those questions which we would ask in a technology-free exam. Hence these are the questions which we expect students can answer without the help of *any* calculator or computer.
- The calculation skills needed to answer the questions from pot –T should be mandatory from school year 8, or starting from the school year in which they are taught. The students are supposed to *maintain* these calculation skills throughout the remaining school years (and, hopefully, beyond school) hence teachers may assess them *at any time*.
- The third pot, +T (= with technology), contains questions which we would not ask in such an exam. Hence in situations in which such problems would occur, we would allow students to use powerful calculators or computers with CAS for their solution.
- The second pot, ?T, reflects our doubts, our different views, and partly also the inherent difficulties of this topic. We either were divided over the questions which ended up in this pot, or we agreed that we would not or could not put them into one of the other two pots. This pot shows how fuzzy the border line (still) is – at least for us.

Whenever feasible we outlined the spectrum and the border line of a class of questions by providing comparable examples for both –T and +T.

Higher Demands During Teaching and Exercises

The questions we put into –T are those which we would not ask in a technology-free exam – but we would not ask them in a technology-supported exam either: These questions appear sensible only in the context of appropriate problems, but not as isolated questions. Their best use could be to test how well a student can operate a calculator.

The questions we put into –T describe long-term manual skills. In order to reach this goal it certainly would make sense to let the students practice with more demanding examples at some stage.

To some extent it could make sense to let the students practice some of the examples from +T even *without technology*.

Other Important Skills and Abilities

It goes without saying that other important skills and abilities exist in addition to calculation skills. In a CAS teaching and learning environment many of those skills and abilities will keep their importance. Several will become more important. In any case, they are indispensable also (for details see [Heugl 1999]). Examples of such abilities are:-

- finding expressions
- recognizing structures
- testing
- visualizing
- using technology properly
- documenting calculations or problem solutions properly.

The ability to visualize allows a person to make a "proper sweep of the hand" to sketch a graph of, for example, x^2 or $\sin(x)$.

Among all the skills and abilities teachers are supposed to teach in math classes, *calculation skills* have played and will play an important role. We teach them not only for their own sake (if we did, their relevance would be severely challenged by the availability of powerful calculators and computers), but to some extent because they are prerequisites for the attainment of "higher" abilities such as the above mentioned. Therefore the above mentioned and other abilities play a decisive role when judging the importance of calculation skills, hence they were part of our discussions. This is partly documented by some of the annotations we give.

Mathematics Education Will Not Become Simpler!

We do not believe that mathematics education will become simpler – the contrary is true. The suggested lower level of manual skills reflects our believe that CAS will become standard tools for mathematics teaching and leaming. It also reflects what we believe is our realistic approach as to what we want students to know throughout their school career and beyond. A consequence of the new tools is that mathematics becomes more useable and probably more demanding – but definitely not simpler. After the very unfortunate discussion about "7 years of teaching mathematics is enough" in the German and Austrian press some years ago we definitely do not want to create a similar debate about "trivial symbol manipulation is enough." Most important for us is the distinction between the goals "perform an operation" (to some extent this can be delegated to a calculator) and "choose a strategy" (this cannot be done by the calculator.)

It goes without saying that the following exposition has an impact on many aspects of teaching mathematics: the teaching methods, training methods, homework, curricula, the topics we teach, what teachers need to know, etc. We broached these issues but did not elaborate them. Therefore we do not mention them here.

Our Goal: Permanently Available Minimal Calculation Skills

We want to spark off a long overdue discussion about the mathematical, methodological, and administrative consequences of using CAS and other mathematics software for teaching and learning mathematics.

This text is meant to be challenging, maybe even provocative. Let us face the challenges of the new tools and let us take the necessary steps! In particular this demands the willingness to say goodbye to familiar things if we see the necessity for it.

Questions and Classes of Questions

For this article we restrict ourselves to questions for which one could use powerful calculators or computers with CAS.

Arithmetic – long term minimal competence

	–T (no technology)	*?T*	*+T (with technology)*
01	compute $3\cdot 40$		compute $3.2987\cdot 4.1298$
02	compute $\sqrt{81}$		approximate $\sqrt{80}$ to ... digits
03	estimate $\sqrt{80}$		simplify $\sqrt{80}$
04			calculate $\sqrt{11\cdot\sqrt[3]{11}}$
05	factor 15		factor 30

The example $\sqrt{80}$ (and its variants –T03, +T02, and +T03) demonstrates how important and decisive the formulation of a question is for putting it into a certain pot. The less important the manual calculation skill becomes, the more important becomes the appropriate formulation in order to clarify the objective of the question. This becomes even clearer with some of the questions in the next sections. We agreed that the importance of the teaching goal "estimation" goes far beyond the given example (–T03). It is so important that we need to reach it without technology – although it may be useful to use a calculator as a pedagogical tool, for example when testing the quality of an estimation, when computing the error, or when demonstrating the purpose of estimations.

To avoid a misunderstanding we repeat what we said earlier: The questions from pot +T are questions we would not ask in a technology-free exam. We would not ask these questions in a technology-supported exam either, because these questions appear useless as such, their best use might be to test how well a student can operate a calculator.

The questions from +T just require the skill of evaluating an expressions which typically comes from a more complicated problem. In the long term this should be delegated to a calculator. We need to make sure that the students *understand* what these expressions mean. But for testing such an *understanding*, we need different types of questions.

Nevertheless – this is another reminder – it certainly could make sense to use questions from +T in both technology-free and technology-supported "training units." This could be needed in order to make the questions which we put in

Basically our proposals obey the following rule: elementary calculations (such as the factoring of an integer with only two factors, e.g. 15) are an indispensable skill (therefore these questions belong to –T), whereas calculations requiring a repeated application of elementary calculations (such as the factoring of an integer with three or more factors, e.g. 30) may be delegated to a calculator.

Fractions– long term minimal competence

	T (no technology)	*?T*	*+T (with technology)*
01	simplify $\frac{10^2}{5^2}$		simplify $7\cdot\frac{2}{5}:\frac{4}{6}$
02	simplify $\frac{10^2}{10^5}$		simplify $\frac{100x^3y^2}{10xy^5}$
03	simplify $2:\frac{1}{2}$		
04	simplify $\frac{2}{\frac{1}{2}}$		
05	simplify $\frac{5a}{5}$		
06	simplify $\frac{a}{5}\cdot 5$		
07	simplify $\frac{2}{x}\cdot\frac{x}{y}$		simplify $\frac{a}{b}\cdot\frac{b^2}{3ac}$
08			simplify $3x^2:\frac{2x}{5y^3}$
09	simplify $2a-\frac{a}{3}$		simplify $2a-\frac{a}{3}+\frac{a}{7}$
10	simplify $\frac{a}{3}+\frac{a}{7}$		
11	simplify $\frac{5}{x}-\frac{2}{x}$		

12	simplify $\frac{2}{x}-\frac{5}{y}$	simplify $\frac{2}{x}-\frac{x}{5}$	

–T01: Here we want students to see the obvious calculation $\frac{100}{25}=4$. This is not trivial!

–T02: Expressions like this are needed in physics.

–T03: A corresponding alternative question (of higher value) would be: "*Why* is $2:\frac{1}{2}$ equal to 4?" This involves the ability to recognise structures.

We deliberately would not test if the rule $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ was learned by heart. We consider this a *background goal* – by which we mean a goal which does not need to be tested explicitly in a written exam. Learning such a rule by heart only leads to students who stolidly apply it for adding two fractions – instead of using the most often more appropriate approach of computing the least common multiple of the denominators. Equally, $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ is only a background goal for us. Nevertheless, these rules (which also can be generated with a CAS) are an important teaching topic, also because they are good examples of the structuring of mathematical facts.

Expressions: With and Without Parentheses – long term minimal competence

We mentioned above that the formulation of a question is decisive for its value. In the following table we deliberately did without the usual request "expand" and instead requested "eliminate the parentheses." While the first formulation seems to suggest the application of the distributive rule, the second is non-suggestive, which hence increases the value of the question.

	–T (no technology)	*?T*	*+T (with technology)*
01	eliminate parentheses: $a-(b+3)$	eliminate parentheses: $(5+p)^2$	eliminate parentheses: $3a^2(5a-2b)$
02	eliminate parentheses: $2(a+b)$		eliminate parentheses: $(a^2-3b)(-3a+5b^2)$
03	eliminate parentheses: $2(ab)$		eliminate parentheses: $(2a+t)^2$
04	eliminate parentheses: $3(5a-2b)$		eliminate parentheses: $(5+p)^3$
05	eliminate parentheses: $(3+a)(b-7)$		
06	find equivalent forms of: $2a+2b$		
07	simplify $x^2y^2+(xy)^2$		
08	factor $3ab+6ac$		
09	factor x^2-4	factor x^2+4x+4	factor x^2-x-6

–T09: This question is important because it helps to develop the abilities 'deciding' and 'justifying.' Both abilities are needed for sensibly using a calculator's "factor" key or command.

The distributive rule $a\cdot(b+c)=a\cdot b+a\cdot c$ is a background goal here.

We had a long discussion about questions ?T01 and ?T09. Part of our group thought that the ability of recognizing structures needs this calculation skill. On the other hand, the Austrian CAS projects produced some evidence that using technology supports the ability to choose a strategy without requiring the development of corresponding calculation skills.

Linear Equations – long term minimal competence

	T (no technology)	*?T*	*+T (with technology)*
01	solve w.r.t. x: $x-6=0$		
02	solve w.r.t. x: $5-x=2$		
03	solve w.r.t. x: $3x=12$		
04	solve w.r.t. x: $5x-6=15$		solve w.r.t. x: $5x-6=2x+15$
05	solve w.r.t. y: $\frac{y}{3}=5$		solve w.r.t. x: $2x+3=\frac{4}{3}$
06	solve w.r.t. x: $a\cdot x=5$	solve w.r.t. x: $a\cdot x-6=15$	
07	solve w.r.t. x: $x+1=x$	solve w.r.t. x: $2(x+1)=2x$	
08	solve w.r.t. x: $x+1=x+1$	solve w.r.t. x: $2(x+1)=2x+2$	
09	solve w.r.t. t: $s=v\cdot t$	solve w.r.t. x: $K=k\cdot x+F$	
10	solve w.r.t. r: $U=2r\boldsymbol{p}$		
11	solve w.r.t. x: $\lvert x\rvert=1$		

–T06: This example is important, because currently available CAS do not make the necessary case distinction for values of a.

–T11: CAS often produce answers involving the absolute value function. Therefore students should know this function and handle simple applications technology-free.

Quadratic Equations – long term minimal competence

	T (no technology)	*?T*	*+T (with technology)*
01	solve w.r.t. x: $x^2=4$		solve w.r.t. x: $9x^2=4$
02	solve w.r.t. x: $x^2-4=0$		solve w.r.t. x: $9x^2-4=0$
03	solve w.r.t. x: $x^2-x=0$		
04	solve w.r.t. x: $x^2-4x=0$	solve w.r.t. x: $x^2+4x+4=0$	solve w.r.t. x: $2x^2-5x+9=0$
05	solve w.r.t. x: $x^2=a$		
06	solve w.r.t. r: $A=4\boldsymbol{p}r^2$		solve w.r.t. v_0: $x=\frac{1}{2a}\cdot v_0^2$

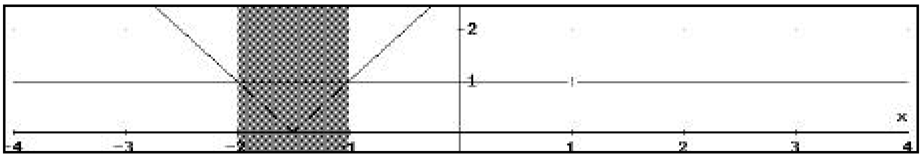
+T04 and ?T04 designate what some teachers may consider the most radical change: we eliminate the formula for the solution of a quadratic equation from the list of indispensable manual skills. However, we keep it as a background goal because of its important role in algebra and the inherent concept of case distinction. The traditional approach of solving quadratic equations with a procedure (by either applying the formula or performing the method of completion of a square) will become extinct (compare [Herget 1996].) For similar reasons the logarithm tables and slide rules disappeared “over night” after arithmetic computations could be delegated to scientific calculators.

Inequalities – long term minimal competence

	T (no technology)	*?T*	*+T (with technology)*
01	for which x is: $x-2<4$	for which x is: $x-2<x+3$	for which x is: $3x+1<2x-1$
02	for which x is: $-2x<4$		for which x is: $\frac{1}{x-1}\le 2$
03	for which x is: $x<x+1$		for which x is: $ax<4$
04	for which x is: $x<x$		
05		for which x is: $\lvert x\rvert<1$	for which x is: $\lvert x-2\rvert<1$

The use of CAS means an obvious shift from calculation to visualization skills as is demonstrated by the following (Derive) screen images.

#4: |2·x + 3| < 1



Differentiation – long term minimal competences

	–T (no technology)	*?T*	*+T (with technology)*
01	differentiate w.r.t. x: $y=x^4$		
02	differentiate w.r.t. x: $y=7x^2+3x+1$		
03	differentiate w.r.t. x: $y=\frac{1}{x^2}$		
04	differentiate w.r.t. x: $y=3$		
05	differentiate w.r.t. x: $y=\sqrt{x}$		
06	differentiate w.r.t. x: $y=\sin x$	differentiate w.r.t. x: $y=x^2+\cos x$	differentiate w.r.t. x: $y=x\sin x$
07		differentiate w.r.t. x: $y=2\cos x$	differentiate w.r.t. x: $y=\sin^2 x$
08		differentiate w.r.t. x: $y=3\sin 2x$	differentiate w.r.t. x: $y=\frac{\sin x}{x}$
09	differentiate w.r.t. x: $y=e^x$	differentiate w.r.t. x: $y=e^{2x}$	differentiate w.r.t. x: $y=2^x$
10	differentiate w.r.t. x: $y=\ln x$		
11	differentiate w.r.t. x: $y=\lvert x\rvert$		

Traditional calculus courses are full of calculation skills. There is a particularly strong demand for change.

Concluding Remark and Request

As mentioned at the beginning, we like to see this paper as an impulse for a broad discussion about which manual calculation skills we should keep demanding and which we can let go. It was not our goal to provide a detailed and unimpeachable pedagogical analysis – we just wanted to give a pragmatic, brief presentation of our current judgement of the complicated issue of manual calculation skills. We deliberately wanted to be provocative and shake the mainstay of traditional mathematics teaching. Let us know what you think.

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Literature

Herget, Wilfried, 1996.: *Rettet die Ideen! – Rettet die Rezepte?* In: Hischer, H. / Weiß, M. (eds.): Rechenfertigkeit und Begriffsbildung – Zu wesentlichen Aspekten des Mathematikunterrichts vor dem Hintergrund von Computeralgebrasystemen. Hildesheim: Franzbecker, pp.156-169.

Herget, Wilfried, 1999: *Wie viel Termumformung braucht der Mensch? – Taschencomputer und Mathematikunterricht.* In: Amelung, Udo (ed.): Der TI-92 im Mathematikunterricht. Pfingsttagung 1998. Zentrale Koordination Lehrerausbildung, ZKL-Texte Nr. 7, Westfälische Wilhelms-Universität, Münster, pp. 3-19.

Heugl, Helmut, 1999: *The necessary fundamental algebraic competence in the age of Computer Algebra Systems.* Proceedings of the 5th ACDCA Summer Academy, 1999, **http://www.acdca.ac.at**.

Kutzler, Bernhard, 1997: *Introduction to the TI-92 (Handheld Computer Algebra)*. Hagenberg: bk teachware, 184 pages, ISBN 3-901769-02-1.

Kutzler, Bernhard, 1998: *Introduction to the TI-89*. Hagenberg: bk teachware, 62 pages, ISBN 3-901769-14-?.

Kutzler, Bernhard, 2000: *Introduzione alla TI-89*. Media Direct Press. 124 Pagine

Kutzler, Bernhard, 1999: *The Algebraic Calculator as a Pedagogical Tool for Teaching Mathematics*. In: Laughbaum E.D. (ed.): Hand-Held Technology in Mathematics and Science Education: A Collection of Papers, Teachers Teaching with Technology Short Course Program @ The Ohio State University, pp. 98-109. Also: **http://www.kutzler.com** .

Kutzler, Bernhard & Kokol-Voljc, Vlasta, 2000: Introduction to Derive 5. Hagenberg: Soft Warehouse Europe. 277 pages.

Kutzler, Bernhard & Kokol-Voljc, Vlasta, 2000: Introduzione a Derive 5. Media Direct SRL – Italia. 277 pages.

Lehmann, Eberhard, 1999a: *Terme im Mathematikunterricht unter Verwendung von Computergrafik und Computeralgebra*, Hannover: Schroedel-Verlag.

Lehmann, Eberhard, 1999b: *Neue Aspekte im Unterricht über Terme durch Einsatz von Computeralgebra-Systemen.* University Bayreuth, Germany.