Problema n. 2p

Partendo dalla relazione

$$\int_0^\infty \frac{Cosh(ux)}{Cosh(\frac{u}{2})}du = \frac{\pi}{Cos(\pi x)}, \qquad |x| < \frac{1}{2},$$

e derivando, (2n-1) volte, rispetto ad x, (n = 1,2,3,…), si ottiene:

$$\int_0^\infty \frac{u^{2n-1}Sinh(ux)}{Cosh(\frac{u}{2})}du = [\frac{\pi}{Cos(\pi x)}]^{(2n-1)} =$$

$$= 2\pi^{2n}(-1)^n(-i)^{-1}\sum_{k\geq 0}(-1)^k(1+2k)^{2n-1}e^{-i\pi x(1+2k)} \qquad \text{(2p).}$$

Ciò premesso, si chiede di dimostrare che:

1) $\frac{1}{2n}\sum_{h=1}^{n}\binom{2n}{2h}2^{2h}(2^{2h}-1)B_{2h} = 1$

dove B_{2h} rappresenta il numero di Bernoulli di indice 2h;

2) $\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}}] =$

$$= \pi^{2n}(-1)^{n-1}\frac{\sqrt{2}}{2}\frac{1}{4^{2n}(2n)!}[\sum_{h=1}^{n}\binom{2n}{2h}4^{4h}(2^{2h}-1)B_{2n} - 4n]$$

Risoluzione

Utilizzando la relazione:

$$\int_0^\infty \frac{Cosh(ux)}{Cosh(u/2)}du = \frac{\pi}{Cos(\pi x)}, \qquad |x| < \frac{1}{2},$$

e derivando, (2n-1) volte, rispetto ad x, (n = 1,2,3,…), e ponendo dopo, x = 0, otteniamo:

$$\lim_{x\to 0}\int_0^\infty \frac{u^{2n-1}Sinh(ux)}{Cosh(u/2)}du = 0;$$

$$[\frac{\pi}{Cos(\pi x)}]^{(2n-1)}_{x=0} = 2\pi[\sum_{k\geq 0}(-)^k e^{-i\pi x(1+2k)}]^{(2n-1)}_{x=0)} = 2\pi\sum_{k\geq 0}(-)^k(-i\pi)^{2n-1}(1+2k)^{2n-1} = 0,$$

da cui: $\sum_{k\geq 0}(-1)^k(1+2k)^{2n-1} = 1 + \sum_{k\geq 1}(-1)^k(1+2k)^{2n-1} = 1 + \sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n-1}\binom{2n-1}{h}(2k)^h =$

$$= 1 + \sum_{k\geq 1}(-1)^k + \sum_{k\geq 1}(-1)^k\sum_{h=1}^{2n-1}\binom{2n-1}{h}(2k)^h = 1 - \frac{1}{2} + \sum_{h=1}^{2n-1}\binom{2n-1}{h}2^h\sum_{k\geq 1}(-1)^k k^h =$$

$$= \frac{1}{2} + \sum_{h=1}^{n}\binom{2n-1}{2h-1}2^{2h-1}\sum_{k\geq 1}(-1)^k k^{2h-1} = \frac{1}{2} - \frac{1}{2}\sum_{h=1}^{n}\binom{2n-1}{2h-1}2^{2h}(2^{2h}-1)\frac{B_{2h}}{2h} = 0, \text{ da cui:}$$

$$\frac{1}{2n}\sum_{h=1}^{n}\binom{2n}{2h}2^{2h}(2^{2h}-1)B_{2h} = 1 \qquad (2p.3)$$

Nel procedimento per ricavare la (2p.3) abbiamo utilizzato le seguenti formule:

$$\sum_{k\geq 1}(-1)^k = -\frac{1}{2}, \sum_{k\geq 1}(-1)^k k^{2h-1} = (2^{2h}-1)(-\frac{B_{2h}}{2h}), \zeta(1-2h) = -\frac{B_{2h}}{2h}, \zeta(-2h)=0,$$

$$.\binom{2n-1}{2h-1}\frac{1}{2h} = \frac{1}{2n}\binom{2n}{2h}$$

La relazione (2p.3) è stata verificata con programma di matematica.

Punto 2

Dalla relazione

$$\lim_{x\to 1/4}\int_0^\infty \frac{u^{2n-1}Sinh(ux)}{Cosh(u/2)}du = [\frac{\pi}{Cos(\pi x)}]_{x=1/4}^{(2n-1)} = 2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]_{x=1/4}^{(2n-1)},$$

troviamo: $\int_0^\infty \frac{u^{2n-1}Sinh(u/4)}{Cosh(u/2)}du = \int_0^\infty u^{2n-1}(e^{u/4}-e^{-u/4})e^{-u/2}\sum_{k\geq 0}(-1)^k e^{-uk}du =$

$$\int_0^\infty u^{2n-1}\sum_{k\geq 0}(-1)^k(e^{-u/4}-e^{-3u/4})e^{-uk}du = (2n-1)!\sum_{k\geq 0}(-1)^k[\frac{1}{(\frac{1}{4}+k)^{2n}} - \frac{1}{(\frac{3}{4}+k)^{2n}}] =$$

$$= (2n-1)!4^{2n}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}}];$$

$$2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]_{x=1/4}^{(2n-1)} = 2\pi\sum_{k\geq 0}(-1)^k(-i\pi)^{2n-1}(1+2k)^{2n-1}e^{-i\pi/4}(-i)^k =$$

$$= 2\pi^{2n}\frac{(-1)^n}{-i}e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n-1};$$

$$\sum_{k\geq 0}(i)^k(1+2k)^{2n-1} = \sum_{k\geq 0}(-1)^k(1+4k)^{2n-1} + \sum_{k\geq 1}(i)^{2k-1}(4k-1)^{2n-1} =$$

$= \sum_{k\geq 0}(-1)^k(1+4k)^{2n-1} + i\sum_{k\geq 1}(-1)^k(1-4k)^{2n-1}$; quindi:

$$2\pi^{2n}\frac{(-1)^n}{-i}e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n-1} = \pi^{2n}(-1)^n\sqrt{2}(1+i)\sum_{k\geq 0}(i)^k(1+2k)^{2n-1} =$$

$$= \pi^{2n}(-1)^n\sqrt{2}(1+i)\ [\sum_{k\geq 0}(-1)^k(1+4k)^{2n-1} + i\sum_{k\geq 1}(-1)^k(1-4k)^{2n-1}]$$

Prendendo la parte reale della precedente, abbiamo:

$$=\pi^{2n}(-1)^n\sqrt{2}\ [\sum_{k\geq 0}(-1)^k(1+4k)^{2n-1}-\sum_{k\geq 1}(-1)^k(1-4k)^{2n-1}]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}\ [1+\sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n-1}\binom{2n-1}{h}(4k)^h-\sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n-1}\binom{2n-1}{h}(-4k)^h]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}\ [1+\sum_{k\geq 1}(-1)^k\sum_{h=1}^{n}\binom{2n-1}{2h-1}(4k)^{2h-1}-\sum_{k\geq 1}(-1)^k\sum_{h=1}^{n}\binom{2n-1}{2h-1}(-4k)^{2h-1}]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}[1+\frac{1}{2}\sum_{h=1}^{n}\binom{2n-1}{2h-1}4^{2h}\sum_{k\geq 1}(-1)^k k^{2h-1}]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}[1+\frac{1}{2}\sum_{h=1}^{n}\binom{2n-1}{2h-1}4^{2h}(2^{2h}-1)(-\frac{B_{2h}}{2h})]=$$

$$=\pi^{2n}(-1)^{n-1}\sqrt{2}[\frac{1}{4n}\sum_{h=1}^{n}\binom{2n}{2h}4^{2h}(2^{2h}-1)B_{2h}-1];$$ pertanto:

$$(2n-1)!4^{2n}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}}-\frac{1}{(3+4k)^{2n}}]=$$

$$=\pi^{2n}(-1)^{n-1}\sqrt{2}[\frac{1}{4n}\sum_{h=1}^{n}\binom{2n}{2h}4^{2h}(2^{2h}-1)B_{2h}-1],$$ da cui:

$$\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}}-\frac{1}{(3+4k)^{2n}}]=$$

$$=\pi^{2n}(-1)^{n-1}\sqrt{2}\ \frac{4^{-2n}}{2(2n)!}[\sum_{h=1}^{n}\binom{2n}{2h}4^{2h}(2^{2h}-1)B_{2h}-4n]. \qquad (2p.4)$$

La relazione (2p.4) è stata verificata con programma di matematica.