Problema n. 3p

Partendo dalla relazione

$$\int_0^\infty \frac{Cosh(ux)}{Cosh(\frac{u}{2})}du = \frac{\pi}{Cos(\pi x)}, \qquad |x| < \frac{1}{2},$$

e derivando, (2n) volte, rispetto ad x, (n = 1,2,3,…), si ottiene:

$$\int_0^\infty \frac{u^{2n}Cosh(ux)}{Cosh(\frac{u}{2})}du = [\frac{\pi}{Cos(\pi x)}]^{(2n)} =$$

$$= 2\pi^{2n+1}(-1)^n \sum_{k\geq 0}(-1)^k (1+2k)^{2n} e^{-i\pi x(1+2k)} \qquad (3p)$$

Ciò premesso, si chiede di dimostrare che:

1) $\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{(-1)^{n-1}\pi^{2n+1}}{2^{2n+2}(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h} - (2n+1)]$;

2) $\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}} + \frac{1}{(3+4k)^{2n+1}}] =$

$$= \frac{(-1)^{n-1}\pi^{2n+1}\sqrt{2}}{4^{2n+1}2(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h} - 2(2n+1)];$$

B_{2h} rappresenta il numero di Bernoulli di indice 2h.

Risoluzione

Punto 1

Utilizzando la relazione:

$$\int_0^\infty \frac{Cosh(ux)}{Cosh(u/2)}du = \frac{\pi}{Cos(\pi x)}, \qquad |x| < \frac{1}{2},$$

e derivando, (2n) volte, rispetto ad x, (n = 1,2,3,…), e ponendo dopo, x = 0, otteniamo:

$$\int_0^\infty \frac{u^{2n}}{Cosh(u/2)}du = [\frac{\pi}{Cos(\pi x)}]_{x=0}^{(2n)} = 2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]_{x=0}^{(2n)};$$

$$\int_0^\infty \frac{u^{2n}}{Cosh(u/2)}du = \int_0^\infty 2u^{2n}e^{-u/2}\sum_{k\geq 0}(-1)^k e^{-uk}du = 2\sum_{k\geq 0}(-1)^k \frac{(2n)!}{(\frac{1}{2}+k)^{2n+1}} =$$

$$= 2^{2n+2}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}};$$

$$2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]_{x=0}^{(2n)} = 2\pi\sum_{k\geq 0}(-1)^k(-i\pi)^{2n}(1+2k)^{2n} = 2\pi^{2n+1}(-1)^n\sum_{k\geq 0}(-1)^k(1+2k)^{2n} =$$

Ricordiamo che:

$$= \sum_{k\geq 0}(-1)^k(1+2k)^{2n} = 1 + \sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n}\binom{2n}{h}(2k)^h = 1 + \sum_{k\geq 1}(-1)^k + \sum_{h=1}^{2n}\binom{2n}{h}2^h\sum_{k\geq 1}(-1)^k k^h =$$

$$=\frac{1}{2}+\sum_{h=1}^{n}\binom{2n}{2h-1}2^{2h-1}\sum_{k\geq 1}(-1)^k k^{2h-1}=\frac{1}{2}-\frac{1}{2(2n+1)}\sum_{h=1}^{n}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}$$

$$2^{2n+2}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}}=2\pi^{2n+1}(-1)^n\,[\frac{1}{2}-\frac{1}{2(2n+1)}\sum_{h\geq 1}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}]=$$

$$=\pi^{2n+1}(-1)^{n-1}\,[\frac{1}{(2n+1)}\sum_{h\geq 1}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}-1]\text{ ,da cui:}$$

$$\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}}=\frac{\pi^{2n+1}(-1)^{n-1}}{2^{2n+2}(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}-(2n+1)] \qquad (3p.1)$$

La formula (3p.1) è stata verificata con un programma di matematica.

Punto 2
Utilizzando la relazione:

$$\int_0^{\infty}\frac{Cosh(ux)}{Cosh(u/2)}du=\frac{\pi}{Cos(\pi x)}, \qquad |x|<\frac{1}{2},$$

e derivando, (2n) volte, rispetto ad x, (n = 1,2,3,…), e ponendo dopo, x = 1/4, otteniamo:

$$\int_0^{\infty}\frac{u^{2n}Cosh(u/4)}{Cosh(u/2)}du=[\frac{\pi}{Cos(\pi x)}]^{(2n)}_{x=1/4}=2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]^{(2n)}_{x=1/4)};$$

$$\int_0^{\infty}\frac{u^{2n}Cosh(u/4)}{Cosh(u/2)}du=\int_0^{\infty}u^{2n}(e^{u/4}+e^{-u/4})e^{-u/2}\sum_{k\geq 0}(-1)^k e^{-uk}du=$$

$$=\sum_{k\geq 0}(-1)^k\int_0^{\infty}u^{2n}(e^{u/4}+e^{-u/4})e^{-u/2}e^{-uk}du=\sum_{k\geq 0}(-1)^k\int_0^{\infty}u^{2n}(e^{-u/4}+e^{-3u/4})e^{-uk}du=$$

$$=(2n)!\sum_{k\geq 0}(-1)^k[\frac{1}{(\frac{1}{4}+k)^{2n+1}}+\frac{1}{(\frac{3}{4}+k)^{2n+1}}]=(2n)!4^{2n+1}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}}+\frac{1}{(3+4k)^{2n+1}}];$$

$$2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]^{(2n)}_{x=1/4)}=2\pi\sum_{k\geq 0}(-1)^k(-i\pi)^{2n}(1+2k)^{2n}e^{-i\pi/4}(-i)^k=$$

$$=2\pi^{2n+1}(-1)^n e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n}\;;$$

ora, $e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n}=\frac{\sqrt{2}}{2}(1-i)[1+\sum_{k\geq 1}(-1)^k(1+4k)^{2n}-i\sum_{k\geq 1}(-1)^k(4k-1)^{2n}]$;

prendendo le parti reali della relazione precedente, abbiamo:

$$\text{Re}[e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n}]=\frac{\sqrt{2}}{2}[1+\sum_{k\geq 1}(-1)^k(1+4k)^{2n}-\sum_{k\geq 1}(-1)^k(1-4k)^{2n}]=$$

$$=\frac{\sqrt{2}}{2}[1+\sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n}\binom{2n}{h}(4k)^h-\sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n}\binom{2n}{h}(-4k)^h]=$$

$$=\frac{\sqrt{2}}{2}[1+\sum_{k\geq 1}(-1)^k\sum_{h=1}^{n}\binom{2n}{2h-1}(4k)^{2h-1}-\sum_{k\geq 1}(-1)^k\sum_{h=0}^{n}\binom{2n}{2h-1}(-4k)^{2h-1}]=$$

$$=\frac{\sqrt{2}}{2}[1+2\sum_{h=1}^{n}\binom{2n}{2h-1}4^{2h-1}\sum_{k\geq 1}(-1)^k k^{2h-1}]=$$

$$=\frac{\sqrt{2}}{2}[1-2\sum_{h=1}^{n}\binom{2n}{2h-1}4^{2h-1}(2^{2h}-1)\frac{B_{2h}}{2h}]=\frac{\sqrt{2}}{2}[1-\frac{1}{2(2n+1)}\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h}];$$

$$(2n)!4^{2n+1}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}}+\frac{1}{(3+4k)^{2n+1}}]=2\pi^{2n+1}(-1)^n e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n}=$$

$$=\pi^{2n+1}(-1)^{n-1}\sqrt{2}[\frac{1}{2(2n+1)}\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h}-1]\text{, da cui:}$$

$$\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}}+\frac{1}{(3+4k)^{2n+1}}]=$$

$$=\frac{(-1)^{n-1}\pi^{2n+1}\sqrt{2}}{4^{2n+1}2(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h}-2(2n+1)] \qquad (3p.2)$$

La relazione (3p.2) è stata verificata con un programma di matematica.