Problema n. 4p

Utilizzando la relazione

$$\int_0^\infty \frac{t^{z-1}dt}{1+t} = \frac{\pi}{Sin\pi z}\text{, } 0 < z < 1,$$

e ponendo z = 1/n, con n > 1, dimostrare che:

$$\sum_{k=1}^{n} k[Cos\frac{2\pi k}{n} - Cos\frac{2\pi(k-1)}{n}] = n$$

Risoluzione

Utilizzando la relazione

$$\int_0^\infty \frac{t^{z-1}dt}{1+t} = \frac{\pi}{Sin\pi z}\text{, } 0 < z < 1,$$

e ponendo z = 1/n, con n > 1, otteniamo:

$$\int_0^\infty \frac{t^{\frac{1}{n}-1}dt}{1+t} = \frac{\pi}{Sin\frac{\pi}{n}} = (t = u^n) = \int_0^\infty \frac{u^{n(\frac{1}{n}-1)} n u^{n-1} du}{1+u^n} = n\int_0^\infty \frac{du}{1+u^n} =$$

$$= n[\int_0^1 \frac{du}{1+u^n} + \int_0^1 \frac{u^{n-2}du}{1+u^n}] = n\int_0^1 \frac{1+u^{n-2}du}{1+u^n}$$

Ora, $\frac{n(1+u^{n-2})}{1+u^n} = n\sum_{k=0}^{n-1} \frac{1+u_k^{n-2}}{u-u_k}\frac{1}{nu_k^{n-1}} = \sum_{k=0}^{n-1}\frac{1+u_k^{n-2}}{u-u_k}\frac{u_k}{u_k^n}$,

dove $u_k^n = -1$, $u_k = e^{i\pi(2k+1)/n}$, k = 0,1,2,3,…(n-1); quindi:

$\frac{n(1+u^{n-2})}{1+u^n} = -\sum_{k=0}^{n-1}(u_k - \frac{1}{u_k})\frac{1}{u-u_k}$; pertanto:

$$\int_0^\infty \frac{t^{\frac{1}{n}-1}dt}{1+t} = \frac{\pi}{Sin\frac{\pi}{n}} = -\sum_{k=0}^{n-1}(u_k - \frac{1}{u_k})[\ln(u-u_k]_0^1 = -\sum_{k=0}^{n-1}(u_k - \frac{1}{u_k})[\ln(1-\frac{1}{u_k}) =$$

$$= -\sum_{k=0}^{n-1}(e^{i\pi(2k+1)/n} - e^{-i\pi(2k+1)/n})\ln(1-e^{-i\pi(2k+1)/n});$$

$$\ln[1-e^{-i\pi(2k+1)/n}] = \ln\{1 - Cos[\pi(2k+1)/n] + iSin[\pi(2n+1)/n]\} =$$

$$= \frac{1}{2}\ln\{(1-Cos[\pi(2k+1)/n])^2 + (Sin[\pi(2n+1)/n])^2\} + iArcTan\frac{Sin[\pi(2k+1)/n]}{1-Cos[\pi(2k+1)/n]};$$

$$iArcTan\frac{Sin[\pi(2k+1)/n]}{1-Cos[\pi(2k+1)/n]} = iArcTan\frac{Cos[\pi(2k+1)/(2n)]}{Sin[\pi(2k+1)/(2n)]} =$$

$$= iArcTan\frac{Sin\{(\pi/2)[1-(2k+1)/n]\}}{Cos\{(\pi/2)[1-(2k+1)/n]\}} = i\frac{\pi}{2}(1-\frac{2k+1}{n})\text{; quindi:}$$

$$\int_0^\infty \frac{t^{\frac{1}{n}-1}dt}{1+t} = \frac{\pi}{Sin\frac{\pi}{n}} = -\sum_{k=0}^{n-1}(e^{i\pi(2k+1)/n} - e^{-i\pi(2k+1)/n})\ln(1-e^{-i\pi(2k+1)/n}),$$

da cui, uguagliando le parti reali, ricaviamo:

$$\frac{\pi}{Sin\frac{\pi}{n}} = \sum_{k=0}^{n-1} 2Sin[\pi(2k+1)/n]\{\frac{\pi}{2}(1-\frac{2k+1}{n})\}$$, quindi:

$$\frac{1}{Sin\frac{\pi}{n}} = \sum_{k=0}^{n-1}(\frac{n-2k-1}{n})Sin[\pi(2k+1)/n]$$, cioè:

$$\sum_{k=0}^{n-1}(n-2k-1)Sin[\pi(2k+1)/n]Sin\frac{\pi}{n} = n$$

Ricordando che : Cos(a+b) - Cos(a-b) = -2SinaSinb, otteniamo:

$$\sum_{k=0}^{n-1}(n-2k-1)(-\frac{1}{2})[Cos\frac{2\pi(k+1)}{n} - Cos\frac{2\pi k}{n}] = n;$$

sostituendo, nella precedente relazione, k a k+1, troviamo:

$$\sum_{k=1}^{n}(n-2k+1)[Cos\frac{2\pi(k-1)}{n} - Cos\frac{2\pi k}{n}] = 2n; \qquad (4p.1)$$

Inoltre, osserviamo che:

$$\sum_{k=1}^{n}[Cos\frac{2\pi(k-1)}{n} - Cos\frac{2\pi k}{n}] = 0$$, e quindi dalla (4p.1), ricaviamo:

$$\sum_{k=1}^{n}(-2k)[Cos\frac{2\pi(k-1)}{n} - Cos\frac{2\pi k}{n}] = 2n$$, cioè:

$$\sum_{k=1}^{n}k[Cos\frac{2\pi k}{n} - Cos\frac{2\pi(k-1)}{n}] = n \qquad (4p.2)$$

La relazione (4p.2) è stata verificata con un programma di matematica.