Problema n. 5c

Partendo dalla relazione:

$$\int_0^\infty \frac{u^{2n} Sinh(ux)}{Sinh(\frac{u}{2})} du = [\pi Tan(\pi x)]^{(2n)}, \qquad |x| < \frac{1}{2},$$

e ponendo, x = ¼, dimostrare che:

$$\sum_{k\geq 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{(-1)^{n-1}\pi^{2n+1}}{2^{2n+2}(2n+1)!}[\sum_{h=1}^{n} \binom{2n+1}{2h} 2^{2h}(2^{2h}-1)B_{2h} - (2n+1)]$$

dove B_{2h} rappresenta il numero di Bernoulli di indice 2h, (n = 1,2,3…).

Risoluzione

Utilizzando la relazione (3.01)

$$\int_0^\infty \frac{u^{2n} Sinh(ux)}{Sinh(u/2)} du = \pi[Tan(\pi x)]^{2n} = \frac{-2\pi}{i}\sum_{k\geq 0}(-1)^k(-2\pi i)^{2n}(1+k)^{2n} e^{-2\pi i x(1+k)},$$

e ponendo, x = ¼, troviamo:

$$\int_0^\infty \frac{u^{2n} Sinh(u/4)}{Sinh(u/2)} du = \int_0^\infty \frac{u^{2n}}{2Cosh(u/4)} du = \int_0^\infty u^{2n} e^{-u/4}\sum_{k\geq 0}(-1)^k e^{-uk/2} du =$$

$$= \sum_{k\geq 0}(-1)^k \frac{(2n)!}{(\frac{1}{4}+\frac{k}{2})^{2n+1}} = 4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}};$$

$$\frac{-2\pi}{i}\sum_{k\geq 0}(-1)^k(-2\pi i)^{2n}(1+k)^{2n} e^{-2\pi i x(1+k)} = 2^{2n+1}\pi^{2n+1}(-1)^n\sum_{k\geq 0}(i)^k(1+k)^{2n} =$$

$$= 2^{2n+1}\pi^{2n+1}(-1)^n[\sum_{k\geq 0}(-1)^k(1+2k)^{2n} + \sum_{k\geq 1}(i)^{2k-1}(2k)^{2n}] = 4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}};$$

uguagliando le parti reali, abbiamo:

$$4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}} = 2^{2n+1}\pi^{2n+1}(-1)^n\sum_{k\geq 0}(-1)^k(1+2k)^{2n};$$

$$\sum_{k\geq 0}(-1)^k(1+2k)^{2n} = 1 + \sum_{k\geq 1}(-1)^k\sum_{h\geq 0}\binom{2n}{h}(2k)^h = 1 + \sum_{k\geq 1}(-1)^k + \sum_{h\geq 1}\binom{2n}{h}2^h\sum_{k\geq 1}(-1)^k k^h;$$

ricordando che $\sum_{k\geq 1}(-1)^k = -\frac{1}{2}$, e che $\zeta(-2h) = 0$, (h = 1,2,3,…), otteniamo:

$$\sum_{k\geq 0}(-1)^k(1+2k)^{2n} = \frac{1}{2} + \sum_{h\geq 1}\binom{2n}{2h-1}2^{2h-1}\sum_{k\geq 1}(-1)^k k^{2h-1} =$$

$$= \frac{1}{2} + \sum_{h\geq 1}\binom{2n}{2h-1}2^{2h-1}(2^{2h}-1)(-\frac{B_{2h}}{2h}) = \frac{1}{2} - \frac{1}{2n+1}\sum_{h\geq 1}\binom{2n+1}{2h}2^{2h-1}(2^{2h}-1)B_{2h};$$ quindi:

$$4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}} = 2^{2n+1}\pi^{2n+1}(-1)^{n-1}[-\frac{1}{2} + \frac{1}{2n+1}\sum_{h\geq 1}\binom{2n+1}{2h}2^{2h-1}(2^{2h}-1)B_{2h}];$$ cioè:

$$\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{\pi^{2n+1}(-1)^{n-1}}{2^{2n+2}(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h} - (2n+1)] \qquad (5c.1)$$

La formula (5c.1) è stata verificata con un programma di matematica.