Problema n. 7c

Partendo dalla relazione:

$$[\pi Tan(\pi x)]^{(2n-1} = \pi^{2n}\sum_{h=0}^{n} a_h [Tan(\pi x)]^{2h}, \qquad |x| < \frac{1}{2},$$

fornire l'espressione che definisce a_h, (h = 0,1,2,3,…,n), in funzione di n, (n = 1,2,3,…).

Risoluzione

Riprendiamo la formula indicata in (2c.2)

$$[\pi Tan(\pi x)]^{(2n-1)} = \pi^{2n}\sum_{h=0}^{n} a_h [Tan(\pi x)]^{2h} = 2^{2n}\pi^{2n}(-1)^n \sum_{k\geq 1}(-1)^k k^{2n-1} e^{-2\pi i x k}, \qquad (7c.1)$$

e poniamo $Tan(\pi x) = t$, da cui: $\pi x = ArcTan(t) = \frac{1}{2i}\ln\frac{1+it}{1-it}$,

che sostituita nella (7c.1) fornisce:

$$\sum_{h=0}^{n} a_h t^{2h} = 2^{2n}(-1)^n \sum_{k\geq 1}(-1)^k k^{2n-1} e^{-k\ln\frac{1+it}{1-it}} = 2^{2n}(-1)^n \sum_{k\geq 1}(-1)^k k^{2n-1}\left(\frac{1-it}{1+it}\right)^k; \qquad (7c.2)$$

derivando la (7c.2), (2h) volte, rispetto a t, e ponendo dopo, t = 0, abbiamo:

$$a_h(2h)! = [2^{2n}(-1)^n \sum_{k\geq 1}(-1)^k k^{2n-1}\left(\frac{1-it}{1+it}\right)^k]_{t=0}^{(2h)};$$

ora, $[(1-it)^k(1+it)^{-k}]_{t=0}^{(2h)} = \lim_{t\to 0}\sum_{j=0}^{2h}\binom{2h}{j}[(1-it)^k]^{(2h-j)}[(1+it)^{-k}]^{(j)} =$

$$= \sum_{j=0}^{2h}\binom{2h}{j}\frac{\Gamma(k+1)(-i)^{2h-j}}{\Gamma(k+1-2h+j)}\frac{\Gamma(k+j)(-i)^j}{\Gamma(k)} = \sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h\frac{\Gamma(k+j)}{\Gamma(k+1-2h+j)} =$$

$$= \sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h(k+j-1)(k+j-2)...(k+j-2h+1);$$

ricordiamo che nello sviluppo delle derivate abbiamo utilizzate le formule seguenti:

$$D_x^{(n)}(a+bx)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)}b^n(a+bx)^{\alpha-n}; \; D_x^{(n)}(a+bx)^{-\beta} = \frac{\Gamma(n+\beta)}{\Gamma(\beta)}(-b)^n(a+bx)^{-\beta-n};$$

inoltre, ricordiamo la ben nota relazione:

x(x-1)(x-2)…(x-n+1) = $\sum_{k=0}^{n} s(n,k)x^k$,

dove s(n,k) rappresentano i numeri di Stirling di prima specie.

Pertanto: $\sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h(k+j-1)(k+j-2)...(k+j-2h+1) =$

$$= \sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)(k+j-1)^u = \sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 0}\binom{u}{p}k^p(j-1)^{u-p};$$

quindi: $a_h(2h)! = 2^{2n}(-1)^n\sum_{k\geq 1}(-1)^k k^{2n}\sum_{j=0}^{2h}\binom{2h}{j}(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 0}\binom{u}{p}k^p(j-1)^{u-p} =$

$$= 2^{2n}(-1)^n(-1)^h\sum_{j=0}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 0}\binom{u}{p}(j-1)^{u-p}\sum_{k\geq 1}(-1)^k k^{2n+p};$$

sappiamo che $\zeta(-2n-2p) = 0$, (n + p intero > 0), e quindi:

$$a_h(2h)! = 2^{2n}(-1)^n(-1)^h\sum_{j=0}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 1}\binom{u}{2p-1}(j-1)^{u-2p+1}\sum_{k\geq 1}(-1)^k k^{2n+2p-1};$$

sappiamo che:

$$\sum_{k\geq 1}(-1)^k k^{2n+2p-1} = (2^{2n+2p}-1)\zeta(1-2n-2p) = (2^{2n+2p}-1)\left(-\frac{B_{2n+2p}}{2n+2p}\right);$$

quindi: $a_h(2h)! =$

$$=2^{2n}(-1)^{n-1}(-1)^h\sum_{j=0}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 1}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}; \quad (7c.3)$$

osserviamo che nello sviluppo della relazione precedente (7c.3), al variare di j, u, p, si presentano valori indeterminati (0^0). Per evitare detti inconvenienti calcoliamo l'espressione precedente per j = 0, per j = 1, e per j > 1.
Così operando, troviamo:

1) per j = 0, $\sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h(k+j-1)(k+j-2)...(k+j-2h+1)=$

$$=(-1)^h k(k-1)(k-2)...(k-2h+1)=(-1)^h\sum_{u=0}^{2h}s(2h,u)k^u;$$

$$[a_h(2h)!]_{j=0}=2^{2n}(-1)^n\sum_{k\geq 1}(-1)^k k^{2n-1}\;(-1)^h\sum_{u=0}^{2h}s(2h,u)k^u=$$

$$=2^{2n}(-1)^n(-1)^h\sum_{u=0}^{2h}s(2h,u)\sum_{k\geq 1}(-1)^k k^{2n+u-1}=$$

$$=2^{2n}(-1)^n(-1)^h\sum_{u=0}^{h}s(2h,2u)\sum_{k\geq 1}(-1)^k k^{2n+2u-1}=$$

$$=2^{2n}(-1)^n(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u}-1)\zeta(1-2n-2u)=$$

$$=2^{2n}(-1)^{n-1}(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}$$

2) per j = 1, $\sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h(k+j-1)(k+j-2)...(k+j-2h+1)=$

$$=2hk(-1)^h(k)(k-1)...(k-2h+2)= 2hk(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)k^u;$$

$$a_h(2h)!]_{j=1} = 2^{2n}(-1)^n\sum_{k\geq 1}(-1)^k k^{2n-1}\;2hk(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)k^u=$$

$$=2^{2n+1}(-1)^n h(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{k\geq 1}(-1)^k k^{2n+u}=$$

$$=2^{2n+1}(-1)^n h(-1)^h\sum_{u=1}^{h}s(2h-1,2u-1)\sum_{k\geq 1}(-1)^k k^{2n+2u-1}; \text{ pertanto:}$$

$$a_h(2h)! = 2^{2n}(-1)^{n-1}(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$

$$+2^{2n+1}(-1)^n h(-1)^h\sum_{u=1}^{h}s(2h-1,2u-1)\sum_{k\geq 1}(-1)^k k^{2n+2u-1}+$$

$$+2^{2n}(-1)^{n-1}(-1)^h\sum_{j=0}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 1}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}.$$

In definitiva, ricaviamo:

$$a_h = \frac{2^{2n}}{(2h)!}(-1)^{n-1}(-1)^h[\sum_{u=0}^{h} s(2h,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$
$$+2h\sum_{u=1}^{h} s(2h-1,2u-1)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$
$$+\sum_{j=2}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1} s(2h-1,u)\sum_{p=1}^{[(u+1)/2]}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}]. \quad (7c.4)$$

Per h = 0, ritroviamo la formula: $a_0 = 2^{2n}(2^{2n}-1)\frac{|B_{2n}|}{2n}$

Per h = n, ricaviamo:

$$a_n = -\frac{2^{2n}}{(2n)!}[\sum_{u=0}^{n} s(2n,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$
$$+2n\sum_{u=1}^{n} s(2n-1,2u-1)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$
$$+\sum_{j=2}^{2n}\binom{2n}{j}\sum_{u=0}^{2n-1} s(2n-1,u)\sum_{p=1}^{[u/2]}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}]=\mathbf{(2n-1)!} \quad (7c.5)$$

Le relazioni (7c.4) e (7c.5) sono state verificate con un programma di matematica.
La verifica della (7c.5) conferma quanto asserito nella soluzione del problema n. 2c, Punto 4.