Problema n. 8c

Partendo dalla relazione:

$$[\pi Tan(\pi x)]^{(2n)} = \pi^{2n+1}\sum_{h=0}^{n} b_h [Tan(\pi x)]^{2h+1}, \qquad |x| < \frac{1}{2},$$

fornire l'espressione che definisce b_h, (h = 0,1,2,3,…,n), in funzione di n, (n = 1,2,3,…).

Risoluzione

Riprendiamo le formule indicate in (3c.1)

$$\pi[Tan(\pi x)]^{2n} = \pi^{2n+1}\sum_{h=0}^{n} b_h [Tan(\pi x)]^{2h+1} = \frac{2\pi}{i}\sum_{k\geq 1}(-1)^k(-2\pi i)^{2n}k^{2n}e^{-2\pi i x k}, \qquad (8c.1)$$

e poniamo $Tan(\pi x) = t$, da cui: $\pi x = ArcTan(t) = \dfrac{1}{2i}\ln\dfrac{1+it}{1-it}$,

che sostituita nella (8c.1) fornisce:

$$\pi^{2n+1}\sum_{h=0}^{n} b_h t^{2h+1} = \frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 1}(-1)^k k^{2n} e^{-k\ln\frac{1+it}{1-it}} =$$

$$= \frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 1}(-1)^k k^{2n}\left(\frac{1-it}{1+it}\right)^k; \qquad (8c.2)$$

derivando la (8c.2), (2h+1) volte, rispetto a t, e ponendo dopo, t = 0, abbiamo:

$$\pi^{2n+1} b_h (2h+1)! = \left[\frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 1}(-1)^k k^{2n}\left(\frac{1-it}{1+it}\right)^k\right]^{(2h+1)}_{t=0};$$

ora, $[(1-it)^k(1+it)^{-k}]^{(2h+1)}_{t=0} = \lim_{t\to 0}\sum_{j=0}^{2h+1}\binom{2h+1}{j}[(1-it)^k]^{(2h+1-j)}[(1+it)^{-k}]^{(j)} =$

$$= \sum_{j=0}^{2h+1}\binom{2h+1}{j}\frac{\Gamma(k+1)(-i)^{2h+1-j}}{\Gamma(k+1-2h-1+j)}\frac{\Gamma(k+j)(-i)^j}{\Gamma(k)} =$$

$$= \sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^h(-i)k\frac{\Gamma(k+j)}{\Gamma(k+j-2h)} =$$

$$= \sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^h(-i)k(k+j-1)(k+j-2)...(k+j-2h) =$$

$$= \sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^h(-i)k\sum_{u=0}^{2h}s(2h,u)(k+j-1)^u =$$

$$= \sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^h(-i)k\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{u}\binom{u}{p}k^p(j-1)^{u-p}; \text{ pertanto:}$$

$b_h(2h+1)! =$

$$= \frac{2^{2n+1}(-1)^n}{i}\sum_{k\geq 1}(-1)^k k^{2n}\sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^h(-i)k\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{u}\binom{u}{p}k^p(j-1)^{u-p} =$$

$$= 2^{2n+1}(-1)^{n-1}(-1)^h\sum_{j=0}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{u}\binom{u}{p}(j-1)^{u-p}\sum_{k\geq 1}(-1)^k k^{2n+p+1} =$$

$$=2^{2n+1}(-1)^{n-1}(-1)^h\sum_{j=0}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}\sum_{k\geq1}(-1)^k k^{2n+2p+1}=$$

$$=2^{2n+1}(-1)^{n+h}\sum_{j=0}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}. \quad (8c.3)$$

Anche qui, osserviamo che nello sviluppo della relazione precedente (8c.3), al variare di j, u, p, si presentano valori indeterminati (0^0).
Per evitare detti inconvenienti calcoliamo l'espressione precedente per j = 0, per j = 1, e per j > 1. Così operando, troviamo:

1) per j = 0, $\sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^h(-i)k(k+j-1)(k+j-2)...(k+j-2h)=$

$$=(-1)^h(-i)k(k-1)(k-2)...(k-2h)=(-1)^h(-i)\sum_{u=0}^{2h+1}s(2h+1,u)k^u$$

$$b_h(2h+1)!]_{j=0}=[\frac{2^{2n+1}(-1)^n}{i}\sum_{k\geq1}(-1)^k k^{2n}\ (-1)^h(-i)\sum_{u=0}^{2h+1}s(2h+1,u)k^u=$$

$$=2^{2n+1}(-1)^{n-1}(-1)^h\sum_{u=0}^{2h+1}s(2h+1,u)\sum_{k\geq1}(-1)^k k^{2n+u}=$$

$$=2^{2n+1}(-1)^{n-1}(-1)^h\sum_{u=0}^{h}s(2h+1,2u+1)\sum_{k\geq1}(-1)^k k^{2n+2u+1}=$$

$$=2^{2n+1}(-1)^n(-1)^h\sum_{u=0}^{h}s(2h+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}$$

2) per j = 1, $\sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^h(-i)k(k+j-1)(k+j-2)...(k+j-2h)=$

$$=(2h+1)(-1)^h(-i)k(k)(k-1)...(k+1-2h)=(2h+1)(-1)^h(-i)k\sum_{u=0}^{2h}s(2h,u)k^u=$$

$$b_h(2h+1)!]_{j=1}=\frac{2^{2n+1}(-1)^n}{i}\sum_{k\geq1}(-1)^k k^{2n}\ (2h+1)(-1)^h(-i)k\sum_{u=0}^{2h}s(2h,u)k^u=$$

$$=2^{2n+1}(-1)^{n-1}(2h+1)(-1)^h\sum_{u=0}^{2h}s(2h,u)\sum_{k\geq1}(-1)^k k^{2n+1+u}=$$

$$=2^{2n+1}(-1)^{n-1}(2h+1)(-1)^h\sum_{u=0}^{h}s(2h,2u)\sum_{k\geq1}(-1)^k k^{2n+1+2u}=$$

$$=2^{2n+1}(-1)^n(2h+1)(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2};$$

Pertanto, ricaviamo:

$$b_h(2h+1)!=2^{2n+1}(-1)^n(-1)^h[\sum_{u=0}^{h}s(2h+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+2^{2n+1}(-1)^n(2h+1)(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+2^{2n+1}(-1)^{n+h}\sum_{j=2}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2},$$

da cui: $b_h=\dfrac{2^{2n+1}}{(2h+1)!}(-1)^n(-1)^h[\sum_{u=0}^{h}s(2h+1,2u+1)(2^{2n+2u+2}-1)\dfrac{B_{2n+2u+2}}{2n+2u+2}+$

$$+(2h+1)\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+\sum_{j=2}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}] \qquad (8c.4)$$

Per h = 0, ritroviamo la formula: $b_0=\dfrac{1}{n+1}(2^{2n+2}-1)2^{2n+1}\left|B_{2n+2}\right|$

Per h = n, ricaviamo:

$$b_n=\frac{2^{2n+1}}{(2n+1)!}[\sum_{u=0}^{n}s(2n+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+(2n+1)\sum_{u=0}^{n}s(2n,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}$$

$$+\sum_{j=2}^{2n+1}\binom{2n+1}{j}\sum_{u=0}^{2n}s(2n,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}] = \mathbf{(2n)!} \qquad (8c.5)$$

Le relazioni (8c.4) e (8c.5) sono state verificate con un programma di matematica.
La verifica della (8c.5) conferma quanto asserito nella soluzione del problema n. 3c, Punto 3.