Problema n. 1p

Partendo dalla relazione dei complementi:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{Sin(\pi z)}, \quad 0 < z < 1,$$

e ponendo $z = \frac{1}{2} - x$, (x, reale),

dimostrare che:

$$\int_0^\infty \frac{Cosh(ux)}{Cosh(\frac{u}{2})} du = \frac{\pi}{Cos(\pi x)}, \qquad |x| < \frac{1}{2}, \tag{1p}$$

Risoluzione

Punto 1

Utilizzando la seguente relazione dei complementi:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{Sin(\pi z)}, \qquad 0 < z < 1$$

e ponendo, $z = \frac{1}{2} - x$, (x reale), otteniamo:

$$\Gamma(x+\frac{1}{2})\Gamma(\frac{1}{2}-x) = \frac{\pi}{Cos(\pi x)}, \qquad |x| < \frac{1}{2} \tag{1p.1}$$

essendo, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-z}$, la ben nota funzione di Eulero di seconda specie.

$$\Gamma(x+\frac{1}{2})\Gamma(\frac{1}{2}-x) = \int_0^1 t^{x-1+1/2}(1-t)^{-x-1+1/2} dt = \int_0^\infty t^{x-1+1/2} \frac{dt}{1+t} = (t = e^u) =$$

$$= \int_{-\infty}^\infty e^{u(x-1+1/2)} \frac{e^u du}{1+e^u} = \int_0^\infty e^{u(x-1+1/2)} \frac{e^u du}{1+e^u} + \int_0^\infty e^{-u(x-1+1/2)} \frac{e^{-u} du}{1+e^{-u}} =$$

$$\int_0^\infty e^{u(x+1/2)} \frac{du}{1+e^u} + \int_0^\infty e^{-u(x+1/2)} \frac{du}{1+e^{-u}} =$$

$$= \int_0^\infty e^{ux} \frac{du}{e^{u/2}+e^{-u/2}} + \int_0^\infty e^{-ux} \frac{du}{e^{u/2}+e^{-u/2}} =$$

$$= \int_0^\infty \frac{Cosh(ux)}{Cosh(u/2)} du = \frac{\pi}{Cos(\pi x)}, \qquad |x| < \frac{1}{2} \tag{1p.2}$$