181. Nonlinear Dynamics and Chaos: an Introduction

Nicola De Nitti
nicoladenitti@gmail.com

I Introduction

Nonlinear dynamics is an active and fascinating discipline that is having a profound effect on a wide variety of topics. Its combination of innovative mathematics and high speed computing has produced new insights into the behaviour of complex systems and has revealed surprising results even in the simplest nonlinear models.
Recently, the ideas of so-called 'chaos theory' have found applications in economics, ecology, population dynamics and sociology. This paper aims to offer a brief overview of nonlinear dynamics and chaos.

II A brief history of chaos

Arguably, Aristotle was already conscious of something similar to what is currently named 'sensitive dependence', since in his writings on methodology and epistemology, he realised that "the minimum initial deviation from the truth is multiplied consequently a thousand fold" (Aristotle, *OTH*, 271b8). Notwithstanding, the investigating how tiny disturbances may grow explosively to generate substantial effects on a physical system's behaviour pattern became a phenomenon of increasingly intensive research starting with a famous paper by Edward Lorenz (1963), in which he documented that a specific meteorological model could exhibit exquisitely sensitive dependence on modest alterations to initial conditions. Previously, the framework for formulating questions related to sensitive dependence had been articulated in 1922 by French mathematician Jacques Hadamard, who argued that any sort of solution showing evidence of sensitive dependence was an indicator of a mathematical model that improperly described its target system.
However, nonlinear dynamics has its origins in the famous 'three body problem', along with the efforts, at the turn of the century, by the brilliant French mathematician and physicist, Henri Poincare, to calculate the movement of a planet around the sun while under the perturbing influence of another nearby planet or moon. In many instances, as expected, the presence of the third body altered the original orbit. Nevertheless, there were also situations in which the planet moved in an extremely erratic way, even to the extent of behaving chaotically.
To have discovered chaos at the heart of a supposedly stable solar system came as a significant surprise. On the other hand, further exploration associated with these ideas had to await the development of new mathematical methods and the development of high-speed computer systems capable of displaying their complex solutions graphically.
Today the application of nonlinear dynamics can be found in almost every branch of science, including systems in which iterations, nonlinear interactions, and the general dependency of each part of the system upon the behaviour of all other parts demands the use of nonlinear differential equations rather than more simple and common linear differential equations.

III The Newtonian paradigm

The assumptions of classical physics are called into question by nonlinear dynamics. Consequently, it is worth examining them in detail.

i. Defining a system

Classical physics assumes that it is possible to focus upon a well-defined system, conceptually isolated from its surroundings, with a characterisation which should not be altered radically over time. For this reason, if the entire nature of the system were to change in an uncontrolled fashion, it would be impossible to be aware of whether it was the same system or something entirely different. Furthermore, based on this theory, it must be possible to separate the internal behaviour of the system from external fluctuations. However, when we enter the nonlinear domain, a great deal of these assumptions are no longer valid.

ii. System description

In physics, the essential features of a natural system can be identified and quantified. It is consequently possible to associate them with mathematical variables (indeed, a system in physics is normally associated with a point in phase space). Moreover, it is assumed that, in principle, it is possible to obtain a complete, experimental description of the system in terms of the numerical values of all its variables, any errors or uncertainties having a negligible implication.

iii. System dynamics

Newton's laws show how it is possible, given an initial point in phase space, to plot out the trajectory of a system for all future time intervals (i.e. given the full specification of a system, it is undoubtedly possible to determine its future behaviour patterns). Any exterior force or perturbation will produce a predictable change while tiny external fluctuations have a negligible effect.
Following from this, it is assumed that the behaviour of the system is orderly, and does not fluctuate erratically or totally change its qualitative nature: when the system begins to deviate from its preassigned or nominal behaviour, it should be possible to exercise control and dampen any unwanted oscillations.

iv. Deviations from the Newtonian paradigm

Where deviations from this well-defined scheme occur, where any sudden qualitative changes in behaviour occur, and where chaotic or wild oscillations are found in a hitherto well behaved system, it is assumed that one can track them down to an external factor, and action can be taken to modify or steer the system back in the right direction. And, while major external alterations may disturb a system, it is assumed that minor fluctuations will produce only very minor changes.

IV Nonlinearities and definition of nonlinear systems

Wherever nonlinearities occur in a system, a situation arises in which one or more of the above assumptions become invalid. While this happens, the whole framework upon which classical physics is based must be called into question, and some new approaches have to be developed.
There are several manifestations of nonlinearity that can frustrate an attempt at analysis of a physical system.

i. Sensitivity to initial conditions and to externalities ('Butterfly effect')

It may not always be possible to pin down a system exactly. There may be, for example, certain unidentified or uncertain factors. The boundary of a system may not be well defined, or the very act of observation and measurement may introduce uncertainties.
For instance, B. Mandelbrot has pointed out that the distribution and number of weather stations has a "lower fractal dimension" than that of any real weather system (i.e. in principle, it is impossible to gather sufficient information to characterise the world's weather). A tiny degree of uncertainty in a linear system does not really influence an attempt at analysis, but for some nonlinear systems these uncertainties increase exponentially. Such systems are infinitely sensitive to their initial conditions: so much so, that the smallest initial fluctuation can quickly swamp the entire system.

Other systems may be infinitely sensitive to externalities – the 'butterfly effect' – so that a tiny fluctuation or perturbation arising in some nearby system will overflow the system. Another aspect of the 'butterfly effect' is that a small periodic effect operating over a long enough period of time might end up dominating the system, whereas large external 'shocks' are dumped out. Consequently, not only will the future of such systems be ambiguous but endeavours at control, or corrective measures, will give unpredictable results.

ii. Sudden changes

Nonlinear systems are characterised by having 'bifurcation-points': regions where the system may suddenly change its qualitative behaviour (in fact, a system that has been behaving in an orderly fashion for a long period of time may suddenly start fluctuating erratically).

Over its life, a nonlinear system can enter a series of quite different behaviours. Furthermore – it must be stressed – these changes need not always be the result of external perturbations, but can be the natural output of the internal dynamics of the system.

iii. Exogenous or endogenous change?

When a system undergoes a sudden dramatic change it is apparently due to some external cause. However, sometimes this major fluctuation or qualitative change has no relation to external circumstances, but is endogenous (i.e. the result of purely internal dynamics). It is of obvious importance to be able to distinguish endogenous from exogenous factors.

iv. Chaotic behaviour

At times, systems enter regions of highly erratic and chaotic behaviour. In such cases it becomes impossible to predict the future behaviour patterns of the system even when based on its entire history.

Although there is no universally accepted mathematical definition of chaos, a commonly adopted explanation states that, for a dynamical system to be classified as chaotic, it must satisfy the following properties:

1. it must be sensitive to initial conditions;
2. it must be topologically mixing;
3. its periodic orbits must be dense.

1. 'Sensitivity to initial conditions' means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Hence, an arbitrarily insignificant perturbation of the present trajectory may lead to significantly different future behaviour. However, it has been shown that the last two properties in the list above actually imply sensitivity to initial conditions and, if analysis is limited to intervals, the second property suggests the other two.

Sensitivity to initial conditions is popularly known as the 'butterfly effect', so named mainly due to the title of a paper submitted by Edward Lorenz in 1972 to the American Association for the Advancement of Science in Washington D.C., entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?* The flapping wing represents a tiny alteration in the initial condition of the system, which may cause a chain of events leading to large-scale phenomena: had the butterfly not flapped its wings, the trajectory of the system could have been immeasurably different.

2. 'Topological mixing' means that the system will evolve over time so that any given region or open set of its phase space will eventually overlap with any other given region.

3. 'Density of periodic orbits' means that every point in the space is approached at an arbitrary distance by periodic orbits.

v. Deterministic chaos

A chaotic system appears perfectly unpredictable in its behaviour. Yet somehow, what is actually known as 'deterministic chaos' appears to exhibit certain regularities. For instance, erratic swings, even though entirely unpredictable, might nevertheless be limited to a specific restricted area, called a 'chaotic or strange attractor'. Therefore, even while the moment-to-moment behaviour of the system is unpredictable, identifying the geometry of the strange attractors offers details about the general range of its behaviour. It is also a matter of debate as to whether a chaotic system should be spoken of as definitely devoid of any order, or as exhibiting a highly complex and not immediately recognisable order. Moreover, such systems may also exhibit 'intermittency', periods of simple order which arise repeatedly out of chaos.

vi. Self-similarity

Chaotic systems have much in common with fractals: indeed their 'strange attractors' have a fractal structure. Likewise, there may be fractal patterns in their dynamics that repeat at different time intervals.

Fractals go beyond the pure mathematics of the concept as the practical uses are just starting to be found. By being able to recognise natural structures with mathematical formulas, it is possible to predict and hypothesise about the future of our environment, species, or many other natural phenomena. John Briggs, in his book entitled *Fractals: The Patterns of Chaos*, mentions many elements in our universe that behave as fractals, reaching the conclusion that having knowledge of the patterns of fractals would make it possible to make better overall micro predictions.

V Examples

The manifestation of nonlinear effects can be discovered in a wide variety of branches of science (e.g. sociology, population dynamics, economics and ecology). However, in every single case, mathematical models may be built. Even if they have the potential for a wide range of behaviour, including stability; gradual growth; persistent oscillations; self-organization; or infinite sensitivity to externalities, obviously, mathematical models are far removed from the real world.

i. Ecology

To analyse a simple case in which nonlinearities occur, it may be interesting to take as an example the effect of increasing carbon dioxide on plant growth: a highly complex issue. In fact, not only will growth rates change but the whole balance of a region will be altered, some species being favoured over others. In turn, the effects of these changing vegetation patterns will feed back into the atmosphere, both directly – in terms of the amount of carbon dioxide that is established by plant-life – but also indirectly, for as the mixes of different vegetation change, so too will the economics as well as the lifestyles of populated regions. As the economy and social structure of a region change, so do its energy demands, which results in different amounts of carbon dioxide being released into the atmosphere.

In addition, attempts to control variations in one part of a cycle may have the effect of magnifying another and even the attempt to isolate a single variable in this whole complex system becomes incredibly complex since a single variable will exhibit the whole range of behaviours from extreme sensitivity to extreme stability, as well as limit cycles, bifurcation points, large oscillations, and possibly even chaotic behaviour. Moreover, this system, by itself, belongs to a much broader system that is inserted into global – and also local – politics. Each of these elements is, in turn, determined by additional factors, including religious as well as ethical values, which are of crucial importance in population growth and attitudes to the environment.

This single example shows how complex a system may be and that a given problem can be sensitive to a wide range of externalities, each of which is related to a range of other factors. Clearly, a completely new science of chaos is required.

ii. Economics

Economics is currently under the scrutiny of experts in the field of nonlinear dynamics and a variety of analyses of short and long term stock market trends have been made. Nevertheless, there are serious

questions to be answered about the very definition of economic systems and about the meaning of their variables. Chaos theory and nonlinear dynamics have been adding weight to those voices that are questioning the basis of economic theory.
The concept of money, for instance, is highly complex and analysts are questioning the notion of economic equilibrium and an intrinsically stable market: “An economic world in which money is introduced in a nontrivial way can be highly complex in its behaviour in theory, just as in reality”, argues Richard Day (University of Southern California). Day himself has already shown that even simple models might give rise to chaotic fluctuations.

VI Order in chaos

The notion that it is the inherent nonlinear dynamics of the market that produce fluctuations, rather than a combination of externalities, may suggest that it might be possible to carry out “micro forecasting”, whereas a more careful analysis will point out that it is impossible to separate endogenous from exogenous causes. Some scholars claim to notice the features of deterministic chaos (i.e. strange attractors) within economic data. Supposing that this is true, it suggests that while economic fluctuations are unpredictable, they will always lie within certain bounds.
Besides, there are certainly suggestions that a degree of self-similarity holds, which is related to fractal structures, and which would indicate that a certain range of behaviour patterns repeats at various time intervals and if this is correct, micro predictions will take into consideration that a random fluctuation will fall within a precise range.
Other analysts are searching for ‘co-operative effects’ (for instance, the choices made in a collective way may give rise to predictable results, or the market itself may manifest a degree of self-organization).

VII Conclusions

The results presented in this paper demonstrate the limitations in describing any nonlinear system, or placing faith in its variables and parameters. Physics normally aims to describe a system and make predictions about general trends, but, as the physicist Richard Feynman puts it, “nature cannot be fooled”, and it is absurd to suppose that mathematical models can cover the wide range of behaviour possible within natural, social and economic systems.
Indeterminacy and related concepts are pervading our existence: in spite of strenuous efforts to reach truth, certainty and precision, both in theoretical and pragmatical matters, mental activity is openly or surreptitiously challenged by indeterminacy, which seems to threaten the inherent order of the natural world.
Attempting to control and correct a natural system will only work within a limited context. Thus, a new science of chaos is required.

Bibliography

ABARBANEL, H. D. I., RABINOVICH, M. I. and SUSHCHIK, M. M., *Introduction to Nonlinear Dynamics for Physicists*, 1993, Singapore, World Scientific.
BAKER, G. L. and GOLLUB, J. B., *Chaotic Dynamics: An Introduction*, 2nd ed., 1996, Cambridge University Press,.
BISHOP, ROBERT, *Chaos,* http://plato.stanford.edu/entries/chaos/
CVITANOVIC, P., *Universality in Chaos: A Reprint Selection*, 2nd ed., 1989, Bristol, Adam Hilger.
DRAZIN, P. G., *Nonlinear Systems*, 1992, Cambridge University Press.
FIELD, M. and GOLUBITSKY, M., Symmetry in Chaos: A Search for Pattern in Mathematics, Art and Nature., 1992, Oxford University Press.
GLEICK, J., *Chaos: Making a New Science,* 1988*,* New York, Penguin.
HILBORN, R. C., *Chaos and Nonlinear Dynamic,* 1994, New York, Oxford University Press.
KAPITANIAK, T. and BISHOP, S. R., The *Illustrated Dictionary of Nonlinear Dynamics and Chaos*, 1998, New York, Wiley.
LORENZ, E. N., *The Essence of Chaos,* 1996, Seattle, University of Washington Press.

MANDELBROT, BENOIT B., *Fractals: Form, Chance, and Dimension*, 1997, San Francisco, W.H. Freeman and Co.
MANDELBROT, BENOIT B., *The Fractal Geometry of Nature*, 1998, New York, W.H. Freeman and Co.
OTT, E. *Chaos in Dynamical Systems*, 1993, New York, Cambridge University Press.
OTT, E., SAUER, T. and YORKE, *J. A. Coping with Chaos: Analysis of Chaotic Data and the Exploitation of Chaotic Systems*, 1994, New York, Wiley.
PEAT, F. D. *Nonlinear Dynamics (Chaos Theory) and its Implications for Policy Planning*, http://www.fdavidpeat.com/bibliography/essays/chaos.htm
PEITGEN, H.-O., JÜRGENS, H. AND SAUPE, D. *Chaos and Fractals: New Frontiers of Science*, 1992, New York, Springer-Verlag.
RASBAND, S. N., *Chaotic Dynamics of Nonlinear Systems,* 1990, New York, Wiley.
SMITH, P., *Explaining Chaos,* 1998, Cambridge, England: Cambridge University Press.
STEWART, I., *Does God Play Dice? The Mathematics of Chaos* , 1990, Blackwell Publishers.
STROGATZ, S. H., *Nonlinear Dynamics and Chaos, with Applications to Physics, Biology, Chemistry, and Engineering*, 1994, Reading, Addison-Wesley.
TABOR, M., *Chaos and Integrability in Nonlinear Dynamics: An Introduction*, 1989, New York, Wiley.
TUFILLARO, N. ABBOTT, T. R. and REILLY, J., *An Experimental Approach to Nonlinear Dynamics and Chaos*, 1992, Redwood City, Addison-Wesley.
WIGGINS, S., *Global Bifurcations and Chaos: Analytical Methods*, 1988, New York, Springer-Verlag.
WIGGINS, S., *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 1990, New York, Springer-Verlag.